

The BC_1 quantum Elliptic model: algebraic forms, hidden algebra $sl(2)$, polynomial eigenfunctions

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Abstract

The potential of the BC_1 quantum elliptic model is a superposition of two Weierstrass functions with doubling of both periods (two coupling constants). The BC_1 elliptic model degenerates to A_1 elliptic model characterized by the Lamé Hamiltonian. It is shown that in the space of BC_1 elliptic invariant, the potential becomes a rational function, while the flat space metric becomes a polynomial. The model possesses the hidden $sl(2)$ algebra for arbitrary coupling constants: it is equivalent to $sl(2)$ -quantum top in three different magnetic fields. It is shown that there exist three one-parametric families of coupling constants for which a finite number of polynomial eigenfunctions (up to a factor) occur.

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The Calogero-Moser-Sutherland models represent a remarkable family of Weyl-invariant integrable systems (with rational, trigonometric/hyperbolic, elliptic potentials), both classical and quantum (see for review and discussions [1]). These models appear in different physical sciences, in particular, in theory of random matrices (see e.g. [2]) and in quantum field theory (see e.g. [3]). In the quantum case, at least some of these models have the outstanding property of (quasi)-exact-solvability when a number of eigenstates can be found explicitly (algebraically). Their gauge-rotated Hamiltonians, written in certain Weyl-invariant variables, are algebraic operators - specifically, differential operators with polynomial coefficients. It is worth noting that the BC_n Calogero-Moser-Sutherland model is a particular case of the Inozemtsev model [4] which is seen as the most general BC_n Weyl-invariant integrable system in \mathbf{R}^n . Both BC_n (elliptic) Calogero-Moser-Sutherland and BC_n (elliptic) Inozemtsev quantum models were extensively studied in [5], [6] and [7] (see references therein), respectively.

Following the formal definition, any one-dimensional dynamics is integrable. Amongst Calogero-Moser-Sutherland models, there exist only two models, A_1 and BC_1 , which describe one-dimensional dynamics (in the case of A_1 , it is the dynamics of the relative motion). A natural question to ask is what distinguishes these two models from all other integrable one-dimensional models. The goal of this paper is to show that both A_1 and BC_1 elliptic quantum systems are equivalent to $sl(2)$ quantum top in a constant magnetic field. They are quasi-exactly-solvable. The spectra of BC_1 elliptic model is also studied.

The BC_1 quantum elliptic model, as it was introduced in Olshanetsky-Perelomov [8], is described by the Hamiltonian

$$\mathcal{H}_{BC_1}^{(e)} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \kappa_2 \wp(2x) + \kappa_3 \wp(x) \equiv -\frac{1}{2}\Delta^{(1)} + V, \quad (1)$$

where $\Delta^{(1)}$ is one-dimensional Laplace operator, $\kappa_{2,3}$ are coupling constants. The Weierstrass function $\wp(x) \equiv \wp(x|g_2, g_3)$ (see e.g. [9]) is defined as

$$(\wp'(x))^2 = 4\wp^3(x) - g_2\wp(x) - g_3 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3), \quad (2)$$

where $g_{2,3}$ are its invariants and $e_{1,2,3}$ are roots, $e_1 + e_2 + e_3 = 0$. If one of the coupling constants vanishes, $\kappa_2(\kappa_3) = 0$, the Hamiltonian becomes (A_1)-Lamé Hamiltonian (see e.g. [10] and references therein). If in the elliptic potential of Eq. (1) the trigonometric limit is taken (one of periods tends to infinity which implies the condition $\Delta \equiv g_2^3 + 27g_3^3 = 0$

holds) the Hamiltonian of BC_1 trigonometric/hyperbolic or generalized Pöschl-Teller model emerges.

Since we will be interested in the general properties of the operator $\mathcal{H}_{BC_1}^{(e)}$, without a loss of generality, we assume that the operator (1) is defined on real line, $x \in \mathbf{R}$ and for the sake of convenience the fundamental domain of the Weierstrass function is rectangular with real period 1 and imaginary period $i \tau$. The discrete symmetry of the Hamiltonian (1) is $\mathbb{Z}_2 \oplus T_r \oplus T_c$. It consists of reflection $\mathbb{Z}_2(x \rightarrow -x)$ which is BC_1 Weyl group and two translations $T_r : x \rightarrow x + 1$ and $T_c : x \rightarrow x + i \tau$ (periodicity). Perhaps, $\mathbb{Z}_2 \oplus T_r \oplus T_c$ can make sense as double-affine BC_1 Weyl group.

We will consider a formal eigenvalue problem

$$\mathcal{H}_{BC_1}^{(e)} \Psi = E \Psi , \quad (3)$$

without posing concrete boundary conditions. It can be immediately checked that (3) has the exact solution

$$\Psi_0 = [\wp'(x)]^\mu , \quad (4)$$

for coupling constants

$$\kappa_2 = 2\mu(\mu - 1) , \quad \kappa_3 = 2\mu(1 + 2\mu) , \quad (5)$$

and μ is an arbitrary parameter, for which the eigenvalue

$$E_0 = 0 .$$

It implies that for parameters (5) the Hamiltonian $\mathcal{H}_{BC_1}^{(e)}$ has one-dimensional invariant subspace, Ψ_0 has the meaning of zero mode, and if $x \in [0, 1]$, the function Ψ_0 (4) is the ground state function (no nodes).

Now let us introduce a new variable,

$$\tau = \wp(x) , \quad (6)$$

cf. [10] and references therein. It is evident that τ is *invariant* with respect to the action of the group $\mathbb{Z}_2 \oplus T_r \oplus T_c$ - double affine BC_1 Weyl group. The first observation is that the potential (1) being written in τ -variable is a rational function,

$$V(\tau) = \frac{\kappa_2 + 4\kappa_3}{4} \tau + \frac{\kappa_2}{16} \frac{12g_2\tau^2 + 36g_3\tau + g_2^2}{4\tau^3 - g_2\tau - g_3}$$

and the ground state function (4) becomes

$$\Psi_0(\tau) = (4\tau^3 - g_2\tau - g_3)^{\frac{\mu}{2}} , \quad (7)$$

(cf. (1)), which is the determinant of the metric with upper indices (see below) to the power $\frac{\mu}{2}$. Making the gauge rotation

$$h^{(e)} = -2(\Psi_0)^{-1} \mathcal{H}_{\text{BC}_1}^{(e)} \Psi_0$$

and changing variable to τ , we arrive at the algebraic operator

$$h^{(e)}(\tau) = \Delta_g(\tau) + \mu(12\tau^2 - g_2)\partial_\tau - \tilde{\kappa}_3\tau \quad (8)$$

where Δ_g is one-dimensional Laplace-Beltrami operator

$$\Delta_g(\tau) = g^{-1/2} \frac{\partial}{\partial \tau} g^{1/2} g^{11} \frac{\partial}{\partial \tau} = g^{11} \frac{\partial^2}{\partial \tau^2} + \frac{g_{,1}^{11}}{2} \frac{\partial}{\partial \tau}$$

with flat metric

$$g^{11} = (4\tau^3 - g_2\tau - g_3) = \frac{1}{g} ,$$

here g is its determinant with upper indices, and

$$\tilde{\kappa}_3 = 2\kappa_3 - 4\mu(1 + 2\mu) \equiv 2n(2n + 1 + 6\mu) .$$

In the explicit form the gauge-rotated operator (8) looks like

$$h^{(e)}(\tau) = (4\tau^3 - g_2\tau - g_3)\partial_\tau^2 + (1 + 2\mu)(6\tau^2 - \frac{g_2}{2})\partial_\tau - 2n(2n + 1 + 6\mu)\tau . \quad (9)$$

It can be easily checked that if parameter n is a non-negative integer, the operator $h^{(e)}(\tau)$ (9) has the invariant subspace

$$\mathcal{P}_n = \langle \tau^p | 0 \leq p \leq n \rangle ,$$

of dimension

$$\dim \mathcal{P}_n = (n + 1) ,$$

namely,

$$h^{(e)} : \mathcal{P}_n \mapsto \mathcal{P}_n .$$

The space \mathcal{P}_n is invariant w.r.t. $1D$ projective (Möbius) transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} .$$

Furthermore, the space \mathcal{P}_n is the finite-dimensional representation space of the algebra $sl(2)$ of the first order differential operators realized as

$$\mathcal{J}^+(n) = \tau^2 \partial_\tau - n\tau, \quad \mathcal{J}^0(n) = \tau \partial_\tau - n, \quad \mathcal{J}^-(n) = \partial_\tau. \quad (10)$$

Hence, the operator (8) can be rewritten in terms of $sl(2)$ -generators

$$\begin{aligned} h^{(e)} &= 4 \mathcal{J}^+(n) \mathcal{J}^0(n) - g_2 \mathcal{J}^0(n) \mathcal{J}^- - g_3 \mathcal{J}^- \mathcal{J}^- \\ &+ 2 \left(4n + 1 + 6\mu \right) \mathcal{J}^+(n) - g_2 \left(n + \frac{1}{2} + \mu \right) \mathcal{J}^-. \end{aligned} \quad (11)$$

Thus, it is $sl(2)$ quantum top in a constant magnetic field. This representation holds for any value of n . Thus, the algebra $sl(2)$ is the hidden algebra of BC_1 elliptic model with arbitrary coupling constants $\kappa_{2,3}$ parametrized as follows

$$\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = (n + 2\mu)(n + 2\mu + 1). \quad (12)$$

If n takes an integer value, the hidden algebra $sl(2)$ appears in finite-dimensional representation, and the operator (9) has finite-dimensional invariant subspace and possesses a number of polynomial eigenfunctions $P_{n,i}(\tau; \mu)$, $i = 1, \dots, (n + 1)$. These polynomials can be called BC_1 Lamé polynomials (of the first kind). If $\mu = 0, 1$ these polynomials degenerate to Lamé polynomials of the first (fourth) kind, respectively. For example, for $n = 0$ at coupling constants (5) (or (12) at $n = 0$),

$$E_{0,1} = 0, \quad P_{0,1} = 1.$$

For $n = 1$ at coupling constants

$$\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = 2(1 + 2\mu)(1 + \mu),$$

the eigenstates are

$$E_{\mp} = \pm(1 + 2\mu)\sqrt{3g_2}, \quad P_{1,\mp} = \tau \mp \frac{1}{2}\sqrt{\frac{g_2}{3}}$$

As a function of g_2 both eigenvalues (eigenfunctions) are branches of double-sheeted Riemann surface. Note that if $\mu = -\frac{1}{2}$ degeneracy occurs: both eigenvalues coincide, they are equal to zero, any linear function is an eigenfunction. If $g_2 = 0$ but $\mu \neq -\frac{1}{2}$, the Jordan cell occurs: both eigenvalues are equal to zero but there exists a single eigenfunction, $P = \tau$. In general, for $n > 1$, polynomial eigenfunctions have a form of a polynomial in τ of degree

n , they (as well as the eigenvalues) are branches of $(n+1)$ -sheeted Riemann surfaces in the parameter g_2 . To summarize, it can be stated that for coupling constants (12) at integer n , the Hamiltonian (1) has $(n+1)$ eigenfunctions of the form

$$\Psi_{n,i} = P_{n,i}(\tau; \mu) \Psi_0, \quad i = 1, \dots, (n+1), \quad (13)$$

where Ψ_0 is given by (4).

It can be checked that the eigenvalue problem (3) has an exact solution other than (4),

$$\Psi_{0,k} = [\wp'(x)]^\mu (\wp(x) - e_k)^{\frac{1}{2}-\mu}, \quad (14)$$

for coupling constants

$$\kappa_2 = 2\mu(\mu-1), \quad \kappa_3 = (1+2\mu)(1-\mu), \quad (15)$$

where μ is an arbitrary parameter, for which the eigenvalue is

$$E_{0,k} = \frac{(4\mu^2 - 1)}{2} e_k,$$

here e_k is the k th root of the Weierstrass function (2). It implies that for parameters (15) the Hamiltonian $\mathcal{H}_{BC_1}^{(e)}$ has one-dimensional invariant subspace.

Making a gauge rotation of the Hamiltonian (1) with subtracted $E_{0,k}$,

$$h_k^{(e)} = -2(\Psi_{0,k})^{-1} (\mathcal{H}_{BC_1}^{(e)} - E_{0,k}) \Psi_{0,k}$$

and changing variable to τ , we arrive at the algebraic operator

$$\begin{aligned} h_k^{(e)}(\tau) &= (\tau - e_k)^{-\frac{1}{2}+\mu} (h^{(e)}(\tau) - 2E_{0,k}) (\tau - e_k)^{\frac{1}{2}-\mu} = \\ &= (4\tau^3 - g_2\tau - g_3)\partial_\tau^2 + 2((5+2\mu)\tau^2 + 2(1-2\mu)e_k(\tau + e_k) - (3-2\mu)\frac{g_2}{4})\partial_\tau \\ &\quad - 2\tilde{\kappa}_3\tau, \end{aligned} \quad (16)$$

(cf. (9)), where e_k is k th root of the Weierstrass function (see (2)), and

$$\tilde{\kappa}_3 = \kappa_3 - (1-\mu)(1+2\mu).$$

It can be checked that if $\tilde{\kappa}_3 = 2n(n-1) + n(2\mu+5)$ and the parameter n takes non-negative integer values, the operator $h_k^{(e)}(\tau)$ has the invariant subspace \mathcal{P}_n . Furthermore,

the operator (16) can be rewritten in terms of $sl(2)$ -generators (10) for any value of n , cf. (11),

$$h_k^{(e)} = 4 \mathcal{J}^+(n) \mathcal{J}^0(n) - g_2 \mathcal{J}^0(n) \mathcal{J}^- - g_3 \mathcal{J}^- \mathcal{J}^- \\ + 2(4n+3+2\mu)\mathcal{J}^+(n) + 4(1-2\mu)e_k(\mathcal{J}^0(n)+n) + 2(2(1-2\mu)e_k^2 - (2n+3-2\mu)\frac{g_2}{4})\mathcal{J}^- . \quad (17)$$

Thus, it is $sl(2)$ quantum top in constant magnetic field.

Hence, the algebra $sl(2)$ is the hidden algebra of BC_1 elliptic model with arbitrary coupling constants $\kappa_{2,3}$ parametrized as follows

$$\kappa_2 = \mu(\mu - 1) , \quad \kappa_3 = 2n^2 + n(3 + 2\mu) + (1 + 2\mu)(1 - \mu) , \quad (18)$$

(cf. (12)). If n takes an integer value, the hidden algebra $sl(2)$ appears in a finite-dimensional representation, the operator (16) has a finite-dimensional invariant subspace and possesses a number of polynomial eigenfunctions $P_{n,i}(\tau; \mu, e_k)$, $i = 1, \dots, (n+1)$ and $k = 1, 2, 3$. These polynomials can be called BC_1 Lamé polynomials (of the second kind). If $\mu = 0, 1$ these polynomials degenerate to Lamé polynomials of the second (third) kind, respectively. For example, for $n = 0$ at couplings (18),

$$E_{0,1} = \frac{(4\mu^2 - 1)}{2} e_k , \quad P_{0,1} = 1 .$$

In general, for $n > 1$, polynomial eigenfunctions have a form of a polynomial in τ of degree n , they (as well as the eigenvalues) are branches of $(n+1)$ -sheeted Riemann surfaces in g_2 . To summarize, it can be stated that for coupling constants (18), and at integer n , the Hamiltonian (1) has $(n+1)$ eigenfunctions of the form

$$\Psi_{n,i;k} = P_{n,i}(\tau; \mu, e_k) \Psi_{0,k} , \quad i = 1, \dots, (n+1) , \quad k = 1, 2, 3 , \quad (19)$$

where $\Psi_{0,k}$ is given by (14).

It can be checked that the eigenvalue problem (3) has one more exact solution other than (4) or (14),

$$\Psi_{0,\tilde{k}} = [\wp'(x)]^\nu [(\wp(x) - e_i)(\wp(x) - e_j)]^{\frac{1}{2}-\nu} , \quad (20)$$

where \tilde{k} is complement to (i, j) , for coupling constants

$$\kappa_2 = 2\nu(\nu - 1) , \quad \kappa_3 = \nu(1 - \nu) , \quad (21)$$

where ν is an arbitrary parameter, for which the eigenvalue is

$$E_{0,\tilde{k}} = \frac{(1 - 2\nu)(3 - 2\nu)}{2} e_k ,$$

here e_k is the \tilde{k} th root of the Weierstrass function (2). It implies that for parameters (21) the Hamiltonian $\mathcal{H}_{BC_1}^{(e)}$ has one-dimensional invariant subspace. If in (20) $\nu = 1 - \mu$ the solution (14) occurs.

Making a gauge rotation of the Hamiltonian (1) with subtracted $E_{0,\tilde{k}}$,

$$h_{\tilde{k}}^{(e)} = -2(\Psi_{0,\tilde{k}})^{-1} (\mathcal{H}_{BC_1}^{(e)} - E_{0,\tilde{k}}) \Psi_{0,\tilde{k}}$$

and changing variable to τ , we arrive at the algebraic operator

$$\begin{aligned} h_{\tilde{k}}^{(e)}(\tau) &= [(\tau - e_i)(\tau - e_j)]^{-\frac{1}{2}+\mu} (h^{(e)}(\tau) - 2E_{0,\tilde{k}}) [(\tau - e_i)(\tau - e_j)]^{\frac{1}{2}-\mu} = \\ &= (4\tau^3 - g_2\tau - g_3)\partial_\tau^2 + 2((7-2\nu)\tau^2 + 2(2\nu-1)e_k(\tau + e_k) - (5+2\nu)\frac{g_2}{4})\partial_\tau \\ &\quad - 2\tilde{\kappa}_3\tau, \end{aligned} \quad (22)$$

(cf. (9)), where e_k is k th root of the Weierstrass function, see (2) and

$$\tilde{\kappa}_3 = \kappa_3 - \nu(3 - 2\nu).$$

It can be checked that if $\tilde{\kappa}_3 = 2n(n-1) + n(7-2\nu)$, and the parameter n takes a non-negative integer value, the operator $h_{\tilde{k}}^{(e)}(\tau)$ has the invariant subspace \mathcal{P}_n . Furthermore, the operator (22) can be rewritten in terms of $sl(2)$ -generators (10) for any value of n , cf. (11),

$$\begin{aligned} h_{\tilde{k}}^{(e)} &= 4\mathcal{J}^+(n)\mathcal{J}^0(n) - g_2\mathcal{J}^0(n)\mathcal{J}^- - g_3\mathcal{J}^-\mathcal{J}^- \\ &+ 2(4n+5-2\nu)\mathcal{J}^+(n) + 4(2\nu-1)e_k(\mathcal{J}^0(n)+n) + 2(2(2\nu-1)e_k^2 - (2n+1+2\nu)\frac{g_2}{4})\mathcal{J}^-. \end{aligned} \quad (23)$$

Thus, it is $sl(2)$ quantum top in constant magnetic field.

Hence, the algebra $sl(2)$ is the hidden algebra of BC_1 elliptic model with arbitrary coupling constants $\kappa_{2,3}$ parametrized as follows

$$\kappa_2 = \nu(\nu-1), \quad \kappa_3 = 2n^2 + n(5-2\nu) + \nu(1-2\nu), \quad (24)$$

(cf. (12), (18)). If n takes an integer value, the hidden algebra $sl(2)$ appears in finite-dimensional representation, and the operator (22) has finite-dimensional invariant subspace \mathcal{P}_n and possesses a number of polynomial eigenfunctions $\tilde{P}_{n,i}(\tau; \nu, e_k)$, $i = 1, \dots, (n+1)$ and $k = 1, 2, 3$. These polynomials can be called BC_1 Lamé polynomials (of the third kind). If $\nu = 0, 1$ these polynomials degenerate to Lamé polynomials of the third (second) kind, respectively. For example, for $n = 0$ at couplings (24),

$$E_{0,1} = \frac{(1-2\nu)(3-2\nu)}{2}e_k, \quad P_{0,1} = 1.$$

In general, for $n > 1$, the polynomial eigenfunctions have a form of a polynomial in τ of degree n , and they (as well as the eigenvalues) are branches of $(n + 1)$ -sheeted Riemann surface in g_2 . To summarize, it can be stated that for coupling constants (24) at integer n the Hamiltonian (1) has $(n + 1)$ eigenfunctions of the form

$$\tilde{\Psi}_{n,i;k} = \tilde{P}_{n,i}(\tau; \nu, e_k) \Psi_{0,k}, \quad i = 1, \dots, (n + 1), \quad k = 1, 2, 3, \quad (25)$$

where $\Psi_{0,\tilde{k}}$ is given by (20).

Observation: Let us construct the operator

$$i_{par}^{(n)}(\tau) = \prod_{j=0}^n (\mathcal{J}^0(n) + j),$$

where $\mathcal{J}^0(n)$ is the Euler-Cartan generator of the algebra $sl(2)$ (10). It can be shown that any algebraic operator $h^{(e)}$ (11), (17), (23) at integer n commutes with $i_{par}^{(n)}(\tau)$,

$$[h^{(e)}(\tau), i_{par}^{(n)}(\tau)] : \mathcal{P}_n \mapsto 0,$$

Hence, $i_{par}^{(n)}(\tau)$ is the particular integral [11] of the BC_1 elliptic model (1).

In this paper we demonstrate that BC_1 elliptic model belongs to one-dimensional quasi-exactly-solvable (QES) problems [12]. However, it is not in the list of known QES problems (see e.g. [13]). We show the existence of three different algebraic forms of the BC_1 Hamiltonian - all of them are the second order polynomial elements of the universal enveloping algebra $U_{sl(2)}$. If this algebra appears in a finite-dimensional representation those elements possess a finite-dimensional invariant subspace. This phenomenon occurs for any of three one-parametric subfamilies of coupling constants for which polynomial eigenfunctions may occur. It is worth noting that a certain algebraic forms for a general BC_n elliptic model were found some time ago in [5, 6].

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